Non-Sylvester Hadamard Matrices

Hadamard matrices are usually introduced through the familiar Walsh/Sylvester construction, where matrices are built recursively from the 2×2 seed:

$H_2 = [[1,1],[1,-1]]$

using the tensor-product recursion:

$H_2\square = [[H\square,H\square],[H\square,-H\square]]$

This produces the well-known power-of-2 Hadamard matrices used throughout signal processing, fast transforms, coding theory, and digital communications.

However, non-Sylvester Hadamard matrices such as H_{12}, H_{20}, H_{28}, and many others appear to have substantially different structural and dynamical properties. Unlike Walsh systems, these matrices are not recursively generated from binary tensor products and therefore lack an inherent dyadic or hypercube organization.

This distinction may have important implications for:

- graph theory,
- additive combinatorics,
- scrambling systems,
- reversible dynamics,
- pseudo-randomness,
- and potentially physics-inspired information transport systems.

The defining Hadamard condition is:

$HH^T = nI$

meaning:

- rows are mutually orthogonal,
- columns are mutually orthogonal,
- entries are restricted to ±1.

After normalization by $1/\sqrt{n}$, Hadamard matrices become orthogonal transforms preserving Euclidean norm and information.

The Walsh/Sylvester family possesses additional structure beyond orthogonality:

- recursive decomposition,
- binary partitioning,

- Walsh harmonic structure,
- hypercube geometry,
- bitwise factorization.

This makes Walsh systems highly compatible with digital computation and FFT-like recursive algorithms.

Non-Sylvester Hadamards generally lack these recursive binary symmetries. Their structure instead emerges from global combinatorial balancing constraints. As a result, they often behave less like recursive signal transforms and more like dense globally coupled interaction systems.

This distinction becomes especially visible in operators of the form:

HD + DH

and:

HD − DH

where:

- H is a Hadamard matrix,
- D is a diagonal ±1 matrix.

These operators naturally admit graph-theoretic interpretations.

For the symmetric operator:

$(HD + DH)_{ij} = H_{ij}(d_i + d_j)$

interaction strength depends on whether vertex states agree. Equal signs reinforce connections while opposite signs suppress them. This behaves like a dynamically gated signed adjacency operator.

For the antisymmetric operator:

$(HD - DH)_{ij} = H_{ij}(d_j - d_i)$

interaction activates primarily across sign boundaries. This resembles:

- edge flow,
- transport operators,
- signed circulation,
- or discrete gauge-like interactions.

The antisymmetric form is closely related to the matrix commutator:

$$[H,D] = HD - DH$$

which measures failure of commutation between global mixing and local sign modulation.

These observations naturally connect Hadamard systems to graph theory.

A Hadamard matrix may be interpreted as a signed complete graph:

- +1 entries represent positive edges,
- −1 entries represent negative edges.

Orthogonality then becomes a global balance condition:
every pair of rows agrees and disagrees equally often.

This places Hadamard matrices close to:

- strongly regular graphs,
- signed graphs,
- expander-like structures,
- finite projective geometries,
- and balanced block designs.

The connection becomes even stronger for non-Sylvester constructions.

Many non-Sylvester Hadamards arise from:

- Paley constructions,
- conference matrices,
- difference sets,
- quadratic residues,
- finite fields.

These constructions are fundamentally combinatorial and graph-theoretic rather than recursive and binary.

Paley constructions are particularly interesting. For prime powers q satisfying:

$$q \equiv 3 \pmod 4$$

Hadamard matrices of order q+1 may be constructed using quadratic residue graphs over finite fields.

This modular arithmetic structure is closely related to:

- primitive roots,
- multiplicative congruential generators (MCGs),
- cyclic difference sets,

- pseudorandom sequence design,
- and spectral distribution theory.

The appearance of $q \equiv 3 \pmod 4$ is not accidental. In such fields:

- -1 is not a quadratic residue,
- symmetry properties of residue graphs change substantially,
- and richer antisymmetric structures emerge.

This same modular behavior appears in high-quality pseudorandom generators and additive combinatorics.

Traditionally, Hadamard theory emphasizes difference sets. For a subset A of a finite group:

$A - A = \{a - b\}$

Difference sets produce globally balanced pairwise interactions and naturally generate orthogonality conditions.

However, the HD + DH operator suggests an additional “sum-set” viewpoint:

$A + A = \{a + b\}$

The symmetric operator depends on sums of diagonal states, while the antisymmetric operator depends on differences. This may indicate that HD+DH and HD−DH encode two distinct combinatorial geometries:

- constructive phase aggregation versus transport,
- coherence versus circulation,
- reinforcement versus boundary flow.

This distinction may not be fully explored in classical Hadamard theory.

The resulting systems resemble:

- signed graph Laplacians,
- discrete wave propagation,
- modular transport systems,
- and reversible scrambling operators.

Compared to Walsh/Sylvester systems, non-Sylvester Hadamards may therefore exhibit:

- less binary resonance,
- weaker hypercube alignment,
- more isotropic mixing,
- denser interaction geometry,
- richer spectra,

- and longer recurrence behavior.

This also aligns with observations from Weyl sequences and modular pseudorandom generators, where avoiding low-order recursive structure often improves mixing quality and suppresses lattice artifacts.

Conceptually, Walsh/Sylvester Hadamards may be viewed as:

- recursively generated orthogonal systems,
- or hypercube reflection operators.

Non-Sylvester Hadamards may instead be viewed as:

- globally balanced signed interaction graphs,
- or dense combinatorial wave-mixing systems.

That distinction may prove important for future work involving:

- reversible neural architectures,
- information-preserving transforms,
- graph dynamics,
- scrambling operators,
- and discrete physical modeling systems.